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IMPLICIT EXTRAPOLATION METHODS FOR VARIABLE COEFFICIENT PROBLEMS

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Abstract. Implicit extrapolation methods for the solution of partial differential equations are based on applying the extrapolation principle *indirectly.* Multigrid tau-extrapolation is a special case of this idea. In the context of multilevel finite element methods, an algorithm of this type can be used to raise the approximation order, even when the meshes are nonuniform or locally refined. The implicit extrapolation multigrid algorithm converges to the solution of a higher order finite element system. This is obtained without explicitly constructing higher order stiffness matrices but by applying extrapolation in a natural form within the algorithm. The algorithm requires only a small change of a basic low order multigrid method.

Key Words. Finite Elements, Extrapolation, Multigrid, Numerical Quadrature.

AMS(MOS) subject classification. 65F10, 65F50, 65N22, 65N50, 65N55.

1. Introduction. Implicit extrapolation is an efficient technique to improve the accuracy of a multilevel solver. When combined with extrapolation, the multilevel principle is not only used as the basis for a fast algebraic solver, but also to increase the approximation order. The basic idea of extrapolation is to exploit discretizations on different levels.

In classical Richardson extrapolation, two or more approximations from different meshes are combined linearly to eliminate the dominating terms of the error expansion. For partial differential equations this has been studied in the context of finite difference discretizations, see e.g. Marchuk and Shaidurov [8] and in the framework of finite elements (FE), see e.g. Blum, Lin, and Rannacher [2]. These techniques are *explicit* extrapolation methods, since they use approximate solutions *directly.*

Here we propose a different approach, where extrapolation is applied *indirectly* to intermediate quantities of the solution process. Such methods are called *implicit* extrapolation techniques. Methods of this type may be related to defect correction, and — if combined with multigrid — to τ-extrapolation, see e.g. Brandt [3], Hackbusch [5], Schaffer [11], or Bernert [1]. However, these methods are mathematically still motivated by expansions of the truncation error, which in turn require uniform meshes. A generalization to locally uniform meshes can e.g. be found in McCormick and Rüde [9].

In Jung and Rüde [6] we have presented an implicit finite element extrapolation technique which is based on extrapolating the quadrature rules used to compute the stiffness matrices. In [6] it has been shown that within the nested

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spaces of a multilevel finite element algorithm, this implicit extrapolation converts an h-hierarchical to a p-hierarchical basis. This improves the approximation order, independent of any uniformity constraints on the mesh and without requiring global asymptotic error expansions. On the other hand, the algorithm presented in [6] is algebraically just a special case of multigrid τ-extrapolation, which differs from the usual multilevel process only by an additional factor appearing in the restriction of the residual. The method is therefore particularly convenient to implement in any given multigrid algorithm.

The analysis of [6] was still restricted to problems with element-wise constant coefficients. In this present paper we will now generalize these results to show that an analogous algorithm can be used for variable coefficients as long as the coefficients are smooth enough to justify higher order approximations at all. The analysis is again based on studying quadrature formulas for the stiffness matrices, and using extrapolation to construct quadrature formulas which are exact for higher order polynomial functions. For variable coefficients, this is now significantly more complicated and our analysis requires nonstandard quadrature rules. These rules and the multilevel algorithm are introduced in detail. The final section presents a numerical example showing the efficiency of the method.

2. The boundary value problem and its finite element discretization . In this paper we consider two-dimensional second order elliptic boundary value problems given in the weak formulation

$$\text{Find } u \in V_0 \text{ such that } \quad a(u,v) = \langle F, v\rangle \quad \text{for all } v \in V_0, \tag{1}$$

with

$$a(u,v) = \int_\Omega (A(x)\nabla_x u, \nabla_x v)\, dx \qquad \text{and} \qquad \langle F, v\rangle = \int_\Omega f v\, dx. \tag{2}$$

Ω is a two-dimensional bounded polygonal domain. The space $V_0 = H_0^1(\Omega)$ is a subspace of the Sobolev space $H^1(\Omega)$, where the functions of V_0 satisfy homogeneous Dirichlet boundary conditions on the boundary $\partial\Omega$. The restriction to this type of boundary conditions is only to keep the exposition as simple as possible. The generalization to somewhat more general boundary conditions is analogous to [6].

Furthermore, we suppose that the 2×2 matrix $A(x) = (a_{ij}(x))_{i,j=1,2}$ is symmetric and positive definite for almost all $x \in \Omega$ with $a_{ij}(x) \in W^2_\infty(\Omega)$. The function f belongs to the space $W^2_q(\Omega)$ with $q \geq 2$. We need these assumptions to obtain a discretization error which is typical for FE discretizations with piecewise quadratic functions and the application of appropriate quadrature rules for the computation of the stiffness matrix and the load vector.

We now discretize (1) by three different finite element spaces. We suppose that two nested triangulations $\mathcal{T}_{l-1}$ and $\mathcal{T}_l$ of the domain Ω are given. The finer triangulation $\mathcal{T}_l$ results from $\mathcal{T}_{l-1}$ by regular refinement, that is by connecting the midpoints of all triangles $\delta_{l-1}^{(r)}$, $r = 1, 2, \ldots, R_{l-1}$, in $\mathcal{T}_{l-1}$. Corresponding to the triangulations $\mathcal{T}_{l-1}$ and $\mathcal{T}_l$ we introduce the finite element spaces

$$
\begin{aligned}
(3) \qquad V_{l-1}^L &= \operatorname{span}\{p_{l-1}^{(i)} \,:\, i = 1, 2, \ldots, N_{l-1}\} \subset V_0, \\
(4) \qquad V_l^L &= V_{l-1}^L \cup \operatorname{span}\{p_l^{(i)} \,:\, i = N_{l-1}+1, \ldots, N_l\} \subset V_0, \\
(5) \qquad V_l^Q &= V_{l-1}^L \cup \operatorname{span}\{q_{l-1}^{(i)} \,:\, i = N_{l-1}+1, \ldots, N_l\} \subset V_0.
\end{aligned}
$$

The trial functions $p_k^{(i)}$, $k = l, l-1$, are continuous and piecewise linear in each triangle of $\mathcal{T}_k$ and they satisfy

$$
\begin{aligned}
p_{l-1}^{(i)}(x^{(j)}) &= \delta_{ij} \quad \text{for } i, j = 1, 2, \ldots, N_{l-1} \\
p_l^{(i)}(x^{(j)}) &= \delta_{ij} \quad \text{for } i, j = N_{l-1}+1, \ldots, N_l.
\end{aligned}
$$

Here $x^{(j)} = (x_1^{(j)}, x_2^{(j)})$ denotes the coordinates of the node $P^{(j)}$ and N_k is the number of nodes of $\mathcal{T}_k$ in Ω. δ_{ij} is the Kronecker symbol.

The functions $q_{l-1}^{(i)}$, $i = N_{l-1}+1, \ldots, N_l$, of (5) are continuous and piecewise *quadratic* in each triangle of $\mathcal{T}_{l-1}$. Again, they satisfy

$$
q_{l-1}^{(i)}(x^{(j)}) = \delta_{ij} \quad \text{for } i, j = N_{l-1}+1, \ldots, N_l\,.
$$

The basis of the space V_l^L we call *h-hierarchical basis* and the basis of the space V_l^Q is called *p-hierarchical basis.*

The finite element subspaces V_{l-1}^L, V_l^L, V_l^Q of (3), (4), and (5), respectively, give rise to the finite element stiffness matrices K_{l-1}^L, K_l^L, and K_l^Q as well as the load vectors $\underline{f}_{l-1}^L$, $\underline{f}_l^L$, and $\underline{f}_l^Q$.

For the computation of the coefficients of the element stiffness matrices and the element load vectors in general we must perform numerical integration. We therefore need an appropriate quadrature rule which guarantees the same FE discretization error as in the case of exact computation of the stiffness matrix and the load vector. To investigate the effect of numerical integration we will use well-known results as e.g. contained in [4]. For the sake of completeness we summarize some of them.

The application of quadrature rules for the computation of the matrix elements and the elements of the load vector results in an approximate bilinear form $\tilde{a}(\tilde{u}, \tilde{v})$ and an approximate right-hand side $\langle \tilde{F}, \tilde{v} \rangle$. Depending on the choice of the quadrature rule and the finite element subspace $\tilde{V}$, i.e. $\tilde{V} = V_{l-1}^L, \tilde{V} = V_l^L$, or $\tilde{V} = V_l^Q$, we will later describe $\tilde{a}(\tilde{u}, \tilde{v})$ in detail.

The approximate bilinear form is called *uniformly* $\tilde{V}$-elliptic, if there exists a constant $\tilde{\alpha} > 0$, $\tilde{\alpha}$ independent of $\tilde{V}$, such that

$$
\tilde{a}(\tilde{v}, \tilde{v}) \geq \tilde{\alpha} ||\tilde{v}||_{1,2,\Omega}^2 \quad \text{for all } \tilde{v} \in \tilde{V}.
$$

Here $|| \cdot ||_{1,2,\Omega}$ denotes the norm in the Sobolev space $H^1(\Omega)$.

Using numerical integration, the boundary value problem (1) is approximated by

$$\text{Find } \tilde{u} \in \tilde{V} \text{ such that } \quad \tilde{a}(\tilde{u}, \tilde{v}) = \langle \tilde{F}, \tilde{v} \rangle \quad \text{for all } \tilde{v} \in \tilde{V}. \tag{6}$$

THEOREM 2.1. (First Lemma of Strang) *Let the approximate bilinear form $\tilde{a}$ of (6) be uniformly $\tilde{V}$-elliptic. Then*

$$||u - \tilde{u}||_{1,2,\Omega} \leq c \left(\inf_{\tilde{v} \in \tilde{V}} \left\{ ||u - \tilde{v}||_{1,2,\Omega} + \sup_{\tilde{w} \in \tilde{V}} \frac{|a(\tilde{v}, \tilde{w}) - \tilde{a}(\tilde{v}, \tilde{w})|}{||\tilde{w}||_{1,2,\Omega}} \right\} + \sup_{\tilde{w} \in \tilde{V}} \frac{|\langle F, \tilde{w} \rangle - \langle \tilde{F}, \tilde{w} \rangle|}{||\tilde{w}||_{1,2,\Omega}} \right)$$

with a constant c which does not depend on the space $\tilde{V}$.

Let the solution $u \in H_0^{s+1}(\Omega)$, $a_{ij} \in W_\infty^s(\Omega)$, $i, j = 1, 2$, $f \in W_q^s(\Omega)$ with $q \geq 2$ and $q > 2/s$, and let the FE subspace $\tilde{V}$ contain piecewise polynomials of degree s, i.e. polynomials of degree s on the triangles of the triangulation. Furthermore, let the quadrature rule be exact for polynomial of degree $2s - 2$ on each triangle. Then the following estimate holds (see also [4])

$$\|u - \tilde{u}\|_{1,2,\Omega} \leq ch^s \Big(|u|_{s+1,2,\Omega} + \sum_{i,j=1}^{2} \|a_{ij}\|_{s,\infty,\Omega} \|u\|_{s+1,2,\Omega} + \|f\|_{s,q,\Omega} \Big) .$$

Here $\| \cdot \|_{s+1,2,\Omega}$ and $| \cdot |_{s+1,2,\Omega}$ denote norms in $H_0^{s+1}(\Omega)$ as well as $\| \cdot \|_{s,q,\Omega}$ is a norm in $W_q^s(\Omega)$.

3. A multigrid algorithm with implicit extrapolation step.

In Jung/Rüde [6] we have studied the convergence properties of a multigrid algorithm with implicit extrapolation step. However, the paper [6] iis restricted to problems with piecewise constant functions $a_{ij}(x)$ and $f(x)$ in $\mathcal{T}_{l-1}$. If such a problem is discretized by *linear* elements, and the multigrid algorithm is combined with (implicit) extrapolation, the iterates converge to the solution given by *quadratic* elements. In this paper we will generalize this result to the case of variable coefficients. It will be shown that the extrapolation algorithm converges to the solution obtained with quadratic elements. In the analysis of this more general case, we will use special nonstandard quadrature rules.

In the following, we will give a brief description of the smoothing procedure and the restriction operator used. Then we formulate the multigrid algorithm and study the convergence behavior.

Numbering the nodes in $\mathcal{T}_l$ such that the nodes which are also in the coarse mesh $\mathcal{T}_{l-1}$ appear first, we induce a block partitioning of the stiffness matrices

$$K_l^L = \begin{pmatrix} K_{l,vv}^L & K_{l,vm}^L \\ K_{l,mv}^L & K_{l,mm}^L \end{pmatrix}, \quad K_l^Q = \begin{pmatrix} K_{l,vv}^Q & K_{l,vm}^Q \\ K_{l,mv}^Q & K_{l,mm}^Q \end{pmatrix}. \tag{7}$$

In the multigrid algorithm we use the following smoothing procedures:

- Pre-smoothing $G_l^V(\underline{u}_l^{(j)}, K_l^L, \underline{f}_l^L)$: Let the initial guess $\underline{u}_l^{(j)} = (\underline{u}_{l,v}^{(j)}, \underline{u}_{l,m}^{(j)})^T$ be given. Set $\underline{u}_{l,v}^{(j+1)} = \underline{u}_{l,v}^{(j)}$ and compute an approximate solution $\underline{z}_{l,m}$ of the system

$$K_{l,mm}^L \underline{z}_{l,m} = \underline{f}_{l,m}^L - K_{l,mv}^L \underline{u}_{l,v}^{(j+1)} - K_{l,mm}^L \underline{u}_{l,m}^{(j)} \tag{8}$$

by means of a linear iterative method starting with the zero vector. We suppose that the error transmission operator of the method is of the type

$$M_{l,m} = I_{l,m} - B_{l,mm}^{-1} K_{l,mm}^L .$$

Then set $\underline{u}_l^{(j+1)} = (\underline{u}_{l,v}^{(j+1)}, \underline{u}_{l,m}^{(j)} + \underline{z}_{l,m})^T$.
- Post-smoothing $G_l^N(\underline{u}_l^{(j)}, K_l^L, \underline{f}_l^L)$: We use the same form of algorithm as for pre-smoothing. However, we suppose that the error transmission operator of the iterative method is of the form $M_{l,m} = I_{l,m} - B_{l,mm}^{-T} K_{l,mm}^L$ such that the overall multigrid operator becomes symmetric.
- We need the *injection operator*

$$I_l^{l-1,inj} : \mathrm{R}^{N_l} \longrightarrow \mathrm{R}^{N_{l-1}}$$

in our algorithm.

Algorithm MG–EX

Let an initial guess $\underline{u}_l^{(k,0)}$ be given.

1. Pre-smoothing:

$$\underline{u}_l^{(k,1)} = G_l^V(\underline{u}_l^{(k,0)}, K_l^L, \underline{f}_l^L) . \tag{9}$$

2. Coarse grid correction:
   (a) Compute the defect

$$\underline{d}_{l-1}^{(k)} = \frac{4}{3}\left(\underline{f}_{l,v}^L - K_{l,vv}^L \underline{u}_{l,v}^{(k,1)} - K_{l,vm}^L \underline{u}_{l,m}^{(k,1)}\right) - \frac{1}{3}\left(\underline{f}_{l-1}^L - K_{l-1}^L I_l^{l-1,inj} \underline{u}_l^{(k,1)}\right) \tag{10}$$

   (b) Solve

$$K_{l-1}^L \underline{w}_{l-1}^{(k)} = \underline{d}_{l-1}^{(k)} , \tag{11}$$

   using μ iteration steps of a usual symmetric multigrid ($(l-1)$–grid) algorithm, starting with the 0 vector and returning an approximate solution $\underline{\tilde{w}}_{l-1}^{(k)}$.

   (c) Correct

$$\underline{u}_l^{(k,2)} = (\underline{u}_{l,v}^{(k,1)} + \underline{\tilde{w}}_{l-1}^{(k)}, \underline{u}_{l,m}^{(k,1)})^T . \tag{12}$$

3. Post-smoothing:

$$\underline{u}_l^{(k,3)} = G_l^N(\underline{u}_l^{(k,2)}, K_l^L, \underline{f}_l^L) \tag{13}$$

and set $\underline{u}_l^{(k+1,0)} = \underline{u}_l^{(k,3)}$.

Taking into consideration the definition of the smoothing procedures and the equivalence of step 2(a) to

$$\underline{d}_{l-1}^{(k)} = \left(\frac{4}{3}\underline{f}_{l,v}^L - \frac{1}{3}\underline{f}_{l-1}^L\right) - \left(\frac{4}{3}K_{l,vv}^L - \frac{1}{3}K_{l-1}^L\right)\underline{u}_{l,v}^{(k,1)} - \frac{4}{3}K_{l,vm}^L\underline{u}_{l,m}^{(k,1)}, \tag{14}$$

we can interpret our algorithm as a usual multigrid algorithm in the h-hierarchical basis to solve the system of equations

$$K_l^{L,ex}\underline{u}_l = \underline{f}_l^{L,ex} \tag{15}$$

with

$$K_l^{L,ex} = \begin{pmatrix} \frac{4}{3}K_{l,vv}^L - \frac{1}{3}K_{l-1}^L & \frac{4}{3}K_{l,vm}^L \\ \frac{4}{3}K_{l,mv}^L & \frac{4}{3}K_{l,mm}^L \end{pmatrix} \quad \text{and} \quad \underline{f}_l^{L,ex} = \begin{pmatrix} \frac{4}{3}\underline{f}_{l,v}^L - \frac{1}{3}\underline{f}_{l-1}^L \\ \frac{4}{3}\underline{f}_{l,m}^L \end{pmatrix}. \tag{16}$$

The main result of this section is that the iterates of the algorithm MG–EX converge to a FE solution which has the same order of discretization error as a FE solution obtained by p-hierarchical FE functions ($p = 2$).

Before we prove this fact, we introduce the quadrature rules that are used to compute the stiffness matrices and load vectors.

To obtain the entries of the stiffness matrices K_{l-1}^L, K_l^L, and K_l^Q, respectively, we have to compute

$$a(\tilde{p}_l^{(j)}, \tilde{p}_l^{(i)}) = \int_\Omega \left(A(x)\nabla_x\tilde{p}_l^{(j)}(x), \nabla_x\tilde{p}_l^{(i)}(x)\right)dx\,, \tag{17}$$

where $\tilde{p}_l^{(i)}$, $\tilde{p}_l^{(j)}$ stand for the functions $p_{l-1}^{(i)}$, $p_{l-1}^{(j)}$, $i,j = 1,\ldots,N_{l-1}$, $p_l^{(i)}$, $p_l^{(j)}$, $i,j = N_{l-1}+1,\ldots,N_l$, in the case of the h-hierarchical basis. In the case of the p-hierarchical basis the functions $\tilde{p}_l^{(i)}$, $\tilde{p}_l^{(j)}$ stand for $p_{l-1}^{(i)}$, $p_{l-1}^{(j)}$, $i,j = 1,\ldots,N_{l-1}$, $q_{l-1}^{(i)}$, $q_{l-1}^{(j)}$, $i,j = N_{l-1}+1,\ldots,N_l$.

First we explain the quadrature rules used for the computation of the matrices K_{l-1}^L and K_l^L. From (17) we obtain for the entries of K_{l-1}^L

$$\int_\Omega \left(A(x)\nabla_x p_{l-1}^{(j)}(x), \nabla_x p_{l-1}^{(i)}(x)\right)dx = \sum_{r\in\omega_{l-1}^{(ij)}} \int_{\delta_{l-1}^{(r)}} \left(A(x)\nabla_x p_{l-1}^{(j)}(x), \nabla_x p_{l-1}^{(i)}(x)\right)dx\,, \tag{18}$$

where

$$\omega_{l-1}^{(ij)} = \left\{r \;:\; p_{l-1}^{(i)} \not\equiv 0 \;\text{ and }\; p_{l-1}^{(j)} \not\equiv 0 \;\text{ on }\; \delta_{l-1}^{(r)}\right\}. \tag{19}$$

We transform the integrals over $\delta_{l-1}^{(r)}$ into integrals over the reference element $\Delta = \{(\xi_1, \xi_2) \,:\, 0 \le \xi_1 \le 1,\, 0 \le \xi_2 \le 1,\, \xi_1 + \xi_2 \le 1\}$. This leads to

$$\begin{aligned}
&\int_{\delta_{l-1}^{(r)}} \Big(A(x) \nabla_x p_{l-1}^{(j)}(x), \nabla_x p_{l-1}^{(i)}(x) \Big)\, dx \\
(20) \quad = \; & \int_\Delta \Big(A(x) (J_{l-1}^{(r)})^{-T} \nabla_\xi p_{l-1}^{(j)}(x(\xi)), (J_{l-1}^{(r)})^{-T} \nabla_\xi p_{l-1}^{(i)}(x(\xi)) \Big)\, |\det J_{l-1}^{(r)}|\, d\xi \\
= \; & \int_\Delta \Big(\bar{B}^{(r)}(x) \nabla_\xi \varphi_{\beta^{(r)}}(\xi), \nabla_\xi \varphi_{\alpha^{(r)}}(\xi) \Big)\, d\xi
\end{aligned}$$

with $\bar{B}^{(r)}(x) = (J_{l-1}^{(r)})^{-1} A(x) (J_{l-1}^{(r)})^{-T} |\det J_{l-1}^{(r)}|$ and $J_{l-1}^{(r)}$ from the transformation

$$(21) \quad \begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} x_1^{(r,2)} - x_1^{(r,1)} & x_1^{(r,3)} - x_1^{(r,1)} \\ x_2^{(r,2)} - x_2^{(r,1)} & x_2^{(r,3)} - x_2^{(r,1)} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} x_1^{(r,1)} \\ x_2^{(r,1)} \end{pmatrix} \\
&= J_{l-1}^{(r)} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} x_1^{(r,1)} \\ x_2^{(r,1)} \end{pmatrix}.
\end{aligned}$$

Here $x_i^{(r,\alpha)}$, $i, j = 1, 2$, $\alpha = 1, 2, 3$, denotes the coordinates of the vertices of the triangle $\delta_{l-1}^{(r)}$, and $\alpha^{(r)}$ as well as $\beta^{(r)}$ are the local numbers of the vertices $P^{(i)}$ and $P^{(j)}$. The linear functions $\varphi_{\alpha^{(r)}}$, $\varphi_{\beta^{(r)}}$, $\alpha^{(r)}$, $\beta^{(r)} = 1, 2, 3$, on the reference element are defined by

$$(22) \qquad \varphi_1(\xi) = 1 - \xi_1 - \xi_2\,, \quad \varphi_2(\xi) = \xi_1\,, \quad \text{and} \quad \varphi_3(\xi) = \xi_2.$$

The following equivalent formulation of (20) is the basis of the application of our quadrature rules.

With the directional derivative

$$(23) \qquad \frac{\partial \varphi}{\partial \xi_s} = \frac{\partial \varphi}{\partial \xi_2} - \frac{\partial \varphi}{\partial \xi_1}$$

we obtain

$$\begin{aligned}
&\int_\Delta \left(\bar{b}_{11}^{(r)} \frac{\partial \varphi_{\beta^{(r)}}}{\partial \xi_1} \frac{\partial \varphi_{\alpha^{(r)}}}{\partial \xi_1} + \bar{b}_{12}^{(r)} \left(\frac{\partial \varphi_{\beta^{(r)}}}{\partial \xi_1} \frac{\partial \varphi_{\alpha^{(r)}}}{\partial \xi_2} + \frac{\partial \varphi_{\beta^{(r)}}}{\partial \xi_2} \frac{\partial \varphi_{\alpha^{(r)}}}{\partial \xi_1} \right) + \bar{b}_{22}^{(r)} \frac{\partial \varphi_{\beta^{(r)}}}{\partial \xi_2} \frac{\partial \varphi_{\alpha^{(r)}}}{\partial \xi_2} \right) d\xi \\
(24) \quad &= \int_\Delta \left(b_{11}^{(r)} \frac{\partial \varphi_{\beta^{(r)}}}{\partial \xi_1} \frac{\partial \varphi_{\alpha^{(r)}}}{\partial \xi_1} + b_{22}^{(r)} \frac{\partial \varphi_{\beta^{(r)}}}{\partial \xi_2} \frac{\partial \varphi_{\alpha^{(r)}}}{\partial \xi_2} + b_{12}^{(r)} \frac{\partial \varphi_{\beta^{(r)}}}{\partial \xi_s} \frac{\partial \varphi_{\alpha^{(r)}}}{\partial \xi_s} \right) d\xi,
\end{aligned}$$

where

$$\begin{aligned}
b_{11}^{(r)}(x(\xi)) &= \bar{b}_{11}^{(r)}(x(\xi)) + \bar{b}_{12}^{(r)}(x(\xi))\,, \qquad b_{22}^{(r)}(x(\xi)) = \bar{b}_{22}^{(r)}(x(\xi)) + \bar{b}_{12}^{(r)}(x(\xi))\,, \\
b_{12}^{(r)}(x(\xi)) &= -\bar{b}_{12}^{(r)}(x(\xi))\,.
\end{aligned}$$

Numerical schemes to evaluate (24) directly have been developed by Rüde [10] and Lyness and Rüde [7]. For the numerical integration of the three terms in (24) we use the following three quadrature rules

$$\int_\Delta v(\xi)d\xi = \text{meas}\Delta\, v(\xi^{(\sigma)}), \qquad \sigma = 1,2,3, \tag{25}$$

with

$$\xi^{(1)} = \left(\frac{1}{2},0\right), \ \xi^{(2)} = \left(0,\frac{1}{2}\right), \text{ and } \xi^{(3)} = \left(\frac{1}{2},\frac{1}{2}\right), \tag{26}$$

respectively. Obviously the quadrature rules in (25) are exact for constant functions v.

The elements of the matrix K_l^L are computed in the same way. We can write the expressions for the computation of the matrix elements in the following formulation

$$\begin{aligned} a(\tilde{p}_l^{(j)},\tilde{p}_l^{(i)}) &= \sum_{r\in\omega_{l-1}^{(ij)}} \int_{\delta_{l-1}^{(r)}} \left(A(x)\nabla_x\tilde{p}_l^{(j)}(x), \nabla_x\tilde{p}_l^{(i)}(x)\right) dx \\ &= \sum_{r\in\omega_{l-1}^{(ij)}} \int_\Delta \left(A(x)(J_{l-1}^{(r)})^{-T}\nabla_\xi\tilde{p}_l^{(j)}(x(\xi)), (J_{l-1}^{(r)})^{-T}\nabla_\xi\tilde{p}_l^{(i)}(x(\xi))\right) |\det J_{l-1}^{(r)}|\, d\xi \\ &= \sum_{r\in\omega_{l-1}^{(ij)}} \sum_{k=1}^{4} \int_{\Delta^{(k)}} \left(\bar{B}^{(r)}(x)\nabla_\xi\varphi_{\beta^{(r)}}(\xi), \nabla_\xi\varphi_{\alpha^{(r)}}(\xi)\right) d\xi\,, \end{aligned} \tag{27}$$

where again $\alpha^{(r)}$, $\beta^{(r)} = 1,2,\dots,6$, are the local numbers of the nodes $P^{(i)}$ and $P^{(j)}$, $\omega_{l-1}^{(ij)} = \left\{r : \tilde{p}_l^{(i)} \not\equiv 0 \text{ and } \tilde{p}_l^{(j)} \not\equiv 0 \text{ on } \delta_{l-1}^{(r)}\right\}$, $\Delta = \cup_{k=1}^4\Delta^{(k)}$ (see also Figure 1), and

$$\begin{aligned} \varphi_1(\xi) &= 1-\xi_1-\xi_2\,, \\ \varphi_2(\xi) &= \xi_1\,, \\ \varphi_3(\xi) &= \xi_2\,, \end{aligned} \qquad \varphi_4(\xi) = \begin{cases} 2\xi_1 & \text{in } \Delta^{(1)} \\ 2-2\xi_1-2\xi_2 & \text{in } \Delta^{(2)} \\ 0 & \text{in } \Delta^{(3)} \\ 1-2\xi_2 & \text{in } \Delta^{(4)} \end{cases}, \tag{28}$$

$$\varphi_5(\xi) = \begin{cases} 0 & \text{in } \Delta^{(1)} \\ 2\xi_2 & \text{in } \Delta^{(2)} \\ 2\xi_1 & \text{in } \Delta^{(3)} \\ 2\xi_1+2\xi_2-1 & \text{in } \Delta^{(4)} \end{cases}, \quad \varphi_6(\xi) = \begin{cases} 2\xi_2 & \text{in } \Delta^{(1)} \\ 0 & \text{in } \Delta^{(2)} \\ 2-2\xi_1-2\xi_2 & \text{in } \Delta^{(3)} \\ 1-2\xi_1 & \text{in } \Delta^{(4)} \end{cases}$$

To compute each integral over $\Delta^{(k)}$ in (27) we use the equivalent formulation of type (24) and a quadrature rule of type (25).

In the case of the p-hierarchical basis, we have to compute the entries of the matrix K_l^Q, i.e. expressions of the form (17), where $\tilde{p}_l^{(i)}, \tilde{p}_l^{(j)}$ stand for the functions $p_{l-1}^{(i)}, p_{l-1}^{(j)}$, $i,j = 1,\dots,N_{l-1}$, $q_{l-1}^{(i)}, q_{l-1}^{(j)}$, $i,j = N_{l-1}+1,\dots,N_l$.

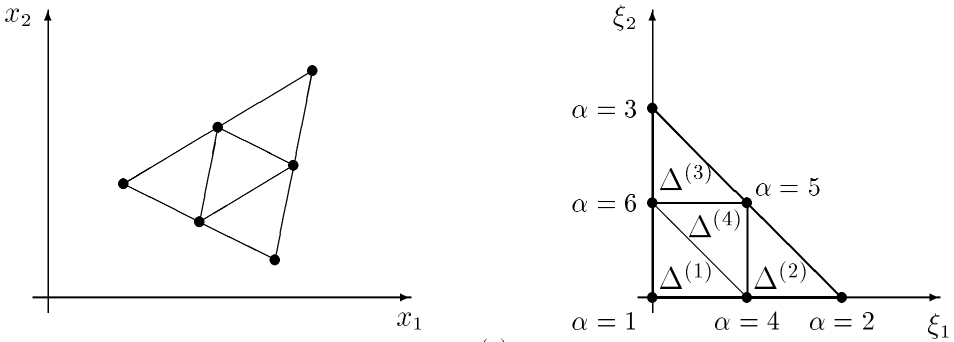


FIG. 1. *An arbitrary triangle* $\delta_{l-1}^{(r)}$ *and the reference element* Δ

Again we get

$$\int_\Omega \left(A(x)\nabla_x \tilde{p}_l^{(j)}(x), \nabla_x \tilde{p}_l^{(i)}(x)\right) dx = \sum_{r\in\omega_{l-1}^{(ij)}} \int_{\delta_{l-1}^{(r)}} \left(A(x)\nabla_x \tilde{p}_l^{(j)}(x), \nabla_x \tilde{p}_l^{(i)}(x)\right) dx\,, \tag{29}$$

with $\omega_{l-1}^{(ij)}$ from (19). After the transformation of the integrals over $\delta_{l-1}^{(r)}$ into integrals over the reference element Δ we obtain the integrals

$$\int_\Delta \left(\bar{B}^{(r)}(x)\nabla_\xi \psi_{\beta^{(r)}}(\xi), \nabla_\xi \psi_{\alpha^{(r)}}(\xi)\right) d\xi\,. \tag{30}$$

The functions $\psi_{\alpha^{(r)}}$ and $\psi_{\beta^{(r)}}$, $\alpha^{(r)}$, $\beta^{(r)} = 1, 2, \ldots, 6$, are defined by

$$\begin{array}{lll} \psi_1(\xi) = 1-\xi_1-\xi_2\,, & \psi_2(\xi) = \xi_1\,, & \psi_3(\xi) = \xi_2\,, \\ \psi_4(\xi) = 4\xi_1(1-\xi_1-\xi_2)\,, & \psi_5(\xi) = 4\xi_1\xi_2\,, & \psi_6(\xi) = 4\xi_2(1-\xi_1-\xi_2)\,. \end{array} \tag{31}$$

The integral (30) we write in the form (24). For the numerical integration of the resulting integrals over Δ we use quadrature rules, which we derive from the quadrature rules (25) by extrapolation. Specifically, we apply for the computation of the first, the second, and the third term the quadrature rules

$$\int_\Delta v(\xi)d\xi \;\approx\; \left\{\frac{4}{3}\left(\frac{1}{4}v(\xi^{(4)}) + \frac{1}{4}v(\xi^{(5)}) + \frac{1}{2}v(\xi^{(6)})\right) - \frac{1}{3}v(\xi^{(1)})\right\} \operatorname{meas}\Delta \tag{32}$$

$$\int_\Delta v(\xi)d\xi \;\approx\; \left\{\frac{4}{3}\left(\frac{1}{4}v(\xi^{(7)}) + \frac{1}{4}v(\xi^{(8)}) + \frac{1}{2}v(\xi^{(9)})\right) - \frac{1}{3}v(\xi^{(2)})\right\} \operatorname{meas}\Delta \tag{33}$$

$$\int_\Delta v(\xi)d\xi \;\approx\; \left\{\frac{4}{3}\left(\frac{1}{4}v(\xi^{(10)}) + \frac{1}{4}v(\xi^{(11)}) + \frac{1}{2}v(\xi^{(12)})\right) - \frac{1}{3}v(\xi^{(3)})\right\} \operatorname{meas}\Delta \tag{34}$$

with $\xi^{(1)}$, $\xi^{(2)}$, $\xi^{(3)}$ from (26) and

$$\begin{array}{lllllllll} \xi^{(4)} &=& (\frac{1}{4},0)\,, & \xi^{(5)} &=& (\frac{3}{4},0)\,, & \xi^{(6)} &=& (\frac{1}{4},\frac{1}{2})\,, \\ \xi^{(7)} &=& (0,\frac{1}{4})\,, & \xi^{(8)} &=& (0,\frac{3}{4})\,, & \xi^{(9)} &=& (\frac{1}{2},\frac{1}{4})\,, \\ \xi^{(10)} &=& (\frac{3}{4},\frac{1}{4})\,, & \xi^{(11)} &=& (\frac{1}{4},\frac{3}{4})\,, & \xi^{(12)} &=& (\frac{1}{4},\frac{1}{4}) \end{array} \tag{35}$$

(see also Figure 2). A simple calculation shows that the quadrature rules (32) – (34) are exact for quadratic functions.

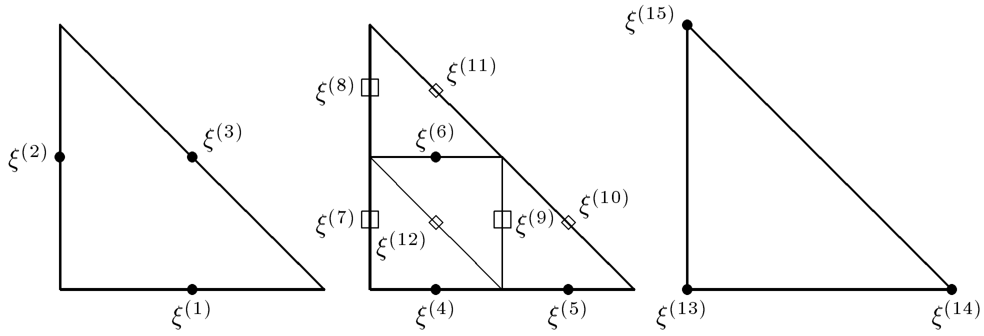


FIG. 2. *Quadrature points of the formulas (25), (32) - (34), (39), and (42)*

Because of the smoothness of the coefficient functions a_{ij} in (2) one can prove that the quadrature rules (25) and (32) – (34) lead for sufficiently small discretization parameters h to a uniformly $\tilde{V}$-elliptic bilinear form $\tilde{a}(.,.)$.

In the following we prove that the extrapolated stiffness matrix in (16) is equal to the stiffness matrix resulting from a discretization with p-hierarchical functions, where we assume that we use the quadrature rules (25) in case of the h-hierarchical basis, and (32) – (34) in case of the p-hierarchical basis.

LEMMA 3.1. *If we compute the element stiffness matrices K_{l-1}^L, K_l^L, and K_l^Q as described above, i.e. by means of the quadrature rules (25) and (32) – (34), the relation*

$$K_l^{L,ex} = K_l^Q \tag{36}$$

holds.

Proof: The proof is based on comparing the matrices $K_l^{L,ex}$ and K_l^Q element by element. The extrapolated stiffness matrix $K_l^{L,ex}$ and the matrix K_l^Q have the block structure

$$K_l^{L,ex} = \begin{pmatrix} \frac{4}{3}K_{l,vv}^L - \frac{1}{3}K_{l-1}^L & \frac{4}{3}K_{l,vm}^L \\ \frac{4}{3}K_{l,mv}^L & \frac{4}{3}K_{l,mm}^L \end{pmatrix}, \quad K_l^Q = \begin{pmatrix} K_{l,vv}^Q & K_{l,vm}^Q \\ K_{l,mv}^Q & K_{l,mm}^Q \end{pmatrix}. \tag{37}$$

The entries of the stiffness matrix K_{l-1}^L are computed using relations (17)–(22) and for the computation of the elements of the matrix K_l^L we use relations (27)–(28).

First, we now prove the identity of the coarse mesh blocks, i.e. $K_{l,vv}^{L,ex} = K_{l,vv}^Q$. Using the quadrature rules of type (25) and the representation (24) with $\alpha^{(r)}$, $\beta^{(r)} = 1, 2, 3$, the elements of the matrix $K_{l,vv}^{L,ex}$ are defined by

$$
\begin{aligned}
K_{l,vv}^{L,ex,(ij)} &= \sum_{r\in\omega_{l-1}^{(ij)}} \Bigg\{ \\
&\frac{4}{3}\sum_{t=4}^{6}\gamma_t^L b_{11}^{(r)}(x(\xi^{(t)}))\frac{\partial\varphi_{\beta^{(r)}}(\xi^{(t)})}{\partial\xi_1}\frac{\partial\varphi_{\alpha^{(r)}}(\xi^{(t)})}{\partial\xi_1} - \frac{1}{3}\gamma_1^L b_{11}^{(r)}(x(\xi^{(1)}))\frac{\partial\varphi_{\beta^{(r)}}(\xi^{(1)})}{\partial\xi_1}\frac{\partial\varphi_{\alpha^{(r)}}(\xi^{(1)})}{\partial\xi_1} \\
&+\frac{4}{3}\sum_{t=7}^{9}\gamma_t^L b_{22}^{(r)}(x(\xi^{(t)}))\frac{\partial\varphi_{\beta^{(r)}}(\xi^{(t)})}{\partial\xi_2}\frac{\partial\varphi_{\alpha^{(r)}}(\xi^{(t)})}{\partial\xi_2} - \frac{1}{3}\gamma_2^L b_{22}^{(r)}(x(\xi^{(2)}))\frac{\partial\varphi_{\beta^{(r)}}(\xi^{(2)})}{\partial\xi_2}\frac{\partial\varphi_{\alpha^{(r)}}(\xi^{(2)})}{\partial\xi_2} \\
&+\frac{4}{3}\sum_{t=10}^{12}\gamma_t^L b_{12}^{(r)}(x(\xi^{(t)}))\frac{\partial\varphi_{\beta^{(r)}}(\xi^{(t)})}{\partial\xi_s}\frac{\partial\varphi_{\alpha^{(r)}}(\xi^{(t)})}{\partial\xi_s} - \frac{1}{3}\gamma_3^L b_{12}^{(r)}(x(\xi^{(3)}))\frac{\partial\varphi_{\beta^{(r)}}(\xi^{(3)})}{\partial\xi_s}\frac{\partial\varphi_{\alpha^{(r)}}(\xi^{(3)})}{\partial\xi_s}\Bigg\}
\end{aligned}
$$

with $\gamma_1^L = \gamma_2^L = \gamma_3^L = \text{meas}\,\Delta$, $\gamma_4^L = \gamma_5^L = \gamma_7^L = \gamma_8^L = \gamma_{10}^L = \gamma_{11}^L = \text{meas}\,\Delta^{(k)}$, and $\gamma_6^L = \gamma_9^L = \gamma_{12}^L = 2\,\text{meas}\,\Delta^{(k)}$.

For the entries of the matrix K_l^Q we get by using relations (29)–(31) and the quadrature rules (32)–(34)

$$
\begin{aligned}
K_{l,vv}^{Q,(ij)} &= \sum_{r\in\omega_{l-1}^{(ij)}} \Bigg\{ \\
&\frac{4}{3}\sum_{t=4}^{6}\gamma_t^Q b_{11}^{(r)}(x(\xi^{(t)}))\frac{\partial\psi_{\beta^{(r)}}(\xi^{(t)})}{\partial\xi_1}\frac{\partial\psi_{\alpha^{(r)}}(\xi^{(t)})}{\partial\xi_1} - \frac{1}{3}\gamma_1^Q b_{11}^{(r)}(x(\xi^{(1)}))\frac{\partial\psi_{\beta^{(r)}}(\xi^{(1)})}{\partial\xi_1}\frac{\partial\psi_{\alpha^{(r)}}(\xi^{(1)})}{\partial\xi_1} \\
&+\frac{4}{3}\sum_{t=7}^{9}\gamma_t^Q b_{22}^{(r)}(x(\xi^{(t)}))\frac{\partial\psi_{\beta^{(r)}}(\xi^{(t)})}{\partial\xi_2}\frac{\partial\psi_{\alpha^{(r)}}(\xi^{(t)})}{\partial\xi_2} - \frac{1}{3}\gamma_2^Q b_{22}^{(r)}(x(\xi^{(2)}))\frac{\partial\psi_{\beta^{(r)}}(\xi^{(2)})}{\partial\xi_2}\frac{\partial\psi_{\alpha^{(r)}}(\xi^{(2)})}{\partial\xi_2} \\
&+\frac{4}{3}\sum_{t=10}^{12}\gamma_t^Q b_{12}^{(r)}(x(\xi^{(t)}))\frac{\partial\psi_{\beta^{(r)}}(\xi^{(t)})}{\partial\xi_s}\frac{\partial\psi_{\alpha^{(r)}}(\xi^{(t)})}{\partial\xi_s} - \frac{1}{3}\gamma_3^Q b_{12}^{(r)}(x(\xi^{(3)}))\frac{\partial\psi_{\beta^{(r)}}(\xi^{(3)})}{\partial\xi_s}\frac{\partial\psi_{\alpha^{(r)}}(\xi^{(3)})}{\partial\xi_s}\Bigg\}
\end{aligned}
$$

with $\gamma_1^Q = \gamma_2^Q = \gamma_3^Q = \text{meas}\,\Delta$, $\gamma_4^Q = \gamma_5^Q = \gamma_7^Q = \gamma_8^Q = \gamma_{10}^Q = \gamma_{11}^Q = 0.25\,\text{meas}\,\Delta$ and $\gamma_6^Q = \gamma_9^Q = \gamma_{12}^Q = 0.5\,\text{meas}\,\Delta$.

If we examine the values of the partial derivatives $\partial\varphi_{\alpha^{(r)}}/\partial\xi_1$, $\partial\varphi_{\alpha^{(r)}}/\partial\xi_2$, $\partial\varphi_{\alpha^{(r)}}/\partial\xi_s$, $\partial\psi_{\alpha^{(r)}}/\partial\xi_1$, $\partial\psi_{\alpha^{(r)}}/\partial\xi_2$, $\partial\psi_{\alpha^{(r)}}/\partial\xi_s$, $\alpha^{(r)} = 1,2,3$, given in Table 1 and the relations $\text{meas}\,\Delta = 4\,\text{meas}\,\Delta^{(k)}$, $k = 1,2,3,4$, $\varphi_{\alpha^{(r)}} = \psi_{\alpha^{(r)}}$, $\alpha^{(r)} = 1,2,3$, we see that

$$
K_{l,vv}^{L,ex,(ij)} = K_{l,vv}^{Q,(ij)} \quad \text{for } i,j = 1,2,\ldots,N_{l-1}\,, \quad \text{i.e. } K_{l,vv}^{L,ex} = K_{l,vv}^{Q}\,.
$$

In the same way we can prove

$$
K_{l,vm}^{L,ex} = K_{l,vm}^{Q}\,, \quad K_{l,mv}^{L,ex} = K_{l,mv}^{Q} \quad \text{and} \quad K_{l,mm}^{L,ex} = K_{l,mm}^{Q}\,.
$$

The symbol "–" in Table 1 means that the partial derivative does not exist in this quadrature point. But we do not need these values for the computation of the matrix elements of $K_l^{L,ex}$. ■

TABLE 1
Values of the partial derivatives in the quadrature points

	$\xi^{(1)}$	$\xi^{(4)}$	$\xi^{(5)}$	$\xi^{(6)}$		$\xi^{(2)}$	$\xi^{(7)}$	$\xi^{(8)}$	$\xi^{(9)}$		$\xi^{(3)}$	$\xi^{(10)}$	$\xi^{(11)}$	$\xi^{(12)}$
$\frac{\partial\varphi_1}{\partial\xi_1}$	-1	-1	-1	-1	$\frac{\partial\varphi_1}{\partial\xi_2}$	-1	-1	-1	-1	$\frac{\partial\varphi_1}{\partial\xi_s}$	0	0	0	0
$\frac{\partial\varphi_2}{\partial\xi_1}$	1	1	1	1	$\frac{\partial\varphi_2}{\partial\xi_2}$	0	0	0	0	$\frac{\partial\varphi_2}{\partial\xi_s}$	-1	-1	-1	-1
$\frac{\partial\varphi_3}{\partial\xi_1}$	0	0	0	0	$\frac{\partial\varphi_3}{\partial\xi_2}$	1	1	1	1	$\frac{\partial\varphi_3}{\partial\xi_s}$	1	1	1	1
$\frac{\partial\varphi_4}{\partial\xi_1}$	—	2	-2	0	$\frac{\partial\varphi_4}{\partial\xi_2}$	—	0	0	-2	$\frac{\partial\varphi_4}{\partial\xi_s}$	—	0	0	-2
$\frac{\partial\varphi_5}{\partial\xi_1}$	—	0	0	2	$\frac{\partial\varphi_5}{\partial\xi_2}$	—	0	0	2	$\frac{\partial\varphi_5}{\partial\xi_s}$	—	2	-2	0
$\frac{\partial\varphi_6}{\partial\xi_1}$	—	0	0	-2	$\frac{\partial\varphi_6}{\partial\xi_2}$	—	2	-2	0	$\frac{\partial\varphi_6}{\partial\xi_s}$	—	0	0	2
$\frac{\partial\psi_4}{\partial\xi_1}$	0	2	-2	0	$\frac{\partial\psi_4}{\partial\xi_2}$	0	0	0	-2	$\frac{\partial\psi_4}{\partial\xi_s}$	0	0	0	-2
$\frac{\partial\psi_5}{\partial\xi_1}$	0	0	0	2	$\frac{\partial\psi_5}{\partial\xi_2}$	0	0	0	2	$\frac{\partial\psi_5}{\partial\xi_s}$	0	2	-2	0
$\frac{\partial\psi_6}{\partial\xi_1}$	0	0	0	-2	$\frac{\partial\psi_6}{\partial\xi_2}$	0	2	-2	0	$\frac{\partial\psi_6}{\partial\xi_s}$	0	0	0	2

Next we discuss the computation of the entries of the load vectors. For the entries of the vector $\underline{f}_{l-1}^{L}$ we get

$$f_{l-1}^{L,(i)} = \langle F, p_{l-1}^{(i)}\rangle = \sum_{r\in\omega_{l-1}^{(i)}} \int_{\delta_{l-1}^{(r)}} f(x)p_{l-1}^{(i)}(x)\,dx = \sum_{r\in\omega_{l-1}^{(i)}} \int_{\Delta} f(x(\xi))\varphi_{\alpha^{(r)}}(\xi)|\det J_{l-1}^{(r)}|\,d\xi \tag{38}$$

with $\omega_{l-1}^{(i)} = \{r\,:\, p_{l-1}^{(i)} \not\equiv 0 \text{ on } \delta_{l-1}^{(r)}\}$.

The integrals over Δ we compute by using the quadrature rule

$$\int_{\Delta} v(\xi)\,d\xi = \left(\frac{1}{3}v(0,0) + \frac{1}{3}v(1,0) + \frac{1}{3}v(0,1)\right) \operatorname{meas}\Delta\,. \tag{39}$$

Obviously, this formula is exact for linear functions v.

The entries of the vector $\underline{f}_{l}^{L}$ are defined by

$$\begin{aligned} f_{l}^{L,(i)} &= \langle F, \tilde{p}_{l}^{(i)}\rangle = \sum_{r\in\omega_{l-1}^{(i)}} \int_{\delta_{l-1}^{(r)}} f(x)\tilde{p}_{l}^{(i)}(x)\,dx = \sum_{r\in\omega_{l-1}^{(i)}} \int_{\Delta} f(x(\xi))\varphi_{\alpha^{(r)}}(\xi)|\det J_{l-1}^{(r)}|\,d\xi \\ &= \sum_{r\in\omega_{l-1}^{(i)}} \sum_{k=1}^{4} \int_{\Delta^{(k)}} f(x(\xi))\varphi_{\alpha^{(r)}}(\xi)|\det J_{l-1}^{(r)}|\,d\xi \end{aligned} \tag{40}$$

with $\tilde{p}_l^{(i)} = p_{l-1}^{(i)}$ for $i = 1, \ldots, N_{l-1}$, $\tilde{p}_l^{(i)} = p_l^{(i)}$ for $i = N_{l-1}+1, \ldots, N_l$ and the functions $\varphi_{\alpha^{(r)}}$ from (28). The integrals over $\Delta^{(k)}$ are computed by a formula of the type (39).

In the case of the p-hierarchical basis, the entries of the load vector $\underline{f}_l^Q$ are given by

$$f_l^{Q,(i)} = \langle F, \tilde{p}_l^{(i)} \rangle = \sum_{r \in \omega_{l-1}^{(i)}} \int_{\delta_{l-1}^{(r)}} f(x) \tilde{p}_l^{(i)}(x)\, dx = \sum_{r \in \omega_{l-1}^{(i)}} \int_{\Delta} f(x(\xi)) \psi_{\alpha^{(r)}}(\xi) |\det J_{l-1}^{(r)}|\, d\xi \tag{41}$$

with $\tilde{p}_l^{(i)} = p_{l-1}^{(i)}$ for $i = 1, \ldots, N_{l-1}$, $\tilde{p}_l^{(i)} = q_{l-1}^{(i)}$ for $i = N_{l-1}+1, \ldots, N_l$, and the functions $\psi_{\alpha^{(r)}}$ from (31). For the computation of the integrals over Δ we use the quadrature rule

$$\int_{\Delta} v(\xi)\, d\xi = \left(\frac{1}{3} v(\xi^{(1)}) + \frac{1}{3} v(\xi^{(2)}) + \frac{1}{3} v(\xi^{(3)}) \right) \operatorname{meas} \Delta\,. \tag{42}$$

with $\xi^{(\sigma)}$, $\sigma = 1, 2, 3$, from (26). This formula is exact for quadratic functions v.

Lemma 3.2. *If the load vectors $\underline{f}_{l-1}^L$, $\underline{f}_l^L$, and $\underline{f}_l^Q$ are defined as described above, then the relation*

$$\underline{f}_l^{L,ex} = \underline{f}_l^Q \tag{43}$$

holds.

Proof: Using the relations (38), (40) and quadrature rules of type (39) for computing the extrapolated load vector $\underline{f}_l^{L,ex}$ (see (16)) as well as relations (41) and (42) for computing the load vector $\underline{f}_l^Q$, the proof follows immediately. ■

A consequence of Lemma 3.1 and Lemma 3.2 is the following Theorem.

Theorem 3.3. *If the extrapolated stiffness matrix $K_l^{L,ex}$ and the extrapolated load vector $\underline{f}_l^{L,ex}$ as well as the stiffness matrix K_l^Q and the load vector $\underline{f}_l^Q$ are computed as described above, then the systems of algebraic FE equations*

$$K_l^{L,ex} \underline{u}_l = \underline{f}_l^{L,ex} \qquad \text{and} \qquad K_l^Q \underline{u}_l = \underline{f}_l^Q \tag{44}$$

have the same solution.

Now we can immediately prove the following convergence theorem for the algorithm MG–EX.

Theorem 3.4. *Under the assumption that the extrapolated stiffness matrix $K_l^{L,ex}$, the extrapolated load vector $\underline{f}_l^{L,ex}$, the stiffness matrix K_l^Q, and the load vector $\underline{f}_l^Q$ are computed as discussed in this section, the following statements hold.*

(i) The iterates of algorithm MG–EX converge to a FE solution which has the same discretization error as a FE solution obtained by a FE discretization with p-hierarchical functions.

(ii) The convergence rate of algorithm MG–EX does not depend on the discretization parameter.

Proof: The statement (i) follows from the interpretation of algorithm MG–EX as a usual multigrid algorithm for solving the system of algebraic equation $K_l^{L,ex}\underline{u}_l = \underline{f}_l^{L,ex}$ and the equivalence of the systems of algebraic equations $K_l^{L,ex}\underline{u}_l = \underline{f}_l^{L,ex}$ and $K_l^{Q}\underline{u}_l = \underline{f}_l^{Q}$.

Statement (ii) we can prove in an analogous way as done for the piecewise constant coefficient case in [6]. ■

Remark: We can also formulate algorithm MG–EX in terms of a piecewise linear nodal basis. All our results are also valid in this case.

4. Numerical results. In this Section we want to confirm our theoretical results by a numerical example. We will illustrate that the iterates of the algorithm MG–EX converge to the FE solution which we would obtain by a discretization of problem (1) with p-hierarchical functions. Furthermore, the numerical example shows that the convergence rate of algorithm MG–EX is independent of the discretization parameter.

All algorithms have been implemented within the multigrid package FEMGP [12]. The computations were performed on a PC 80486 (33 MHz) using the LAHEY-Fortran compiler.

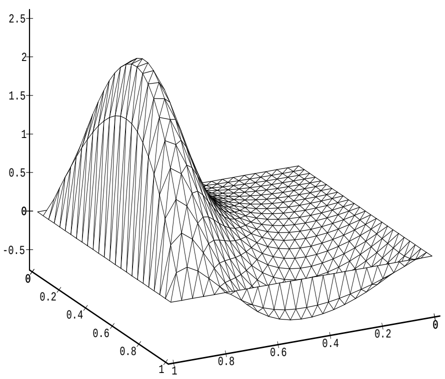
Let us consider the problem (1), where $\Omega = (0,1) \times (0,1)$ and

$$A(x) = \begin{pmatrix} 2 - x_1 + \frac{x_2}{2} & \sin x_1 \sin x_2 \\ \sin x_1 \sin x_2 & e^{x_1+x_2} \end{pmatrix}.$$

The right-hand side $f(x)$ is chosen such that the function

$$u(x) = x_1(1-x_1)\sin x_2 (e^{4x_1^2+x_2^2} - e^{x_1^2+3x_2^2})$$

is the exact solution of problem (1). Figure 3 shows plots of this exact solution.



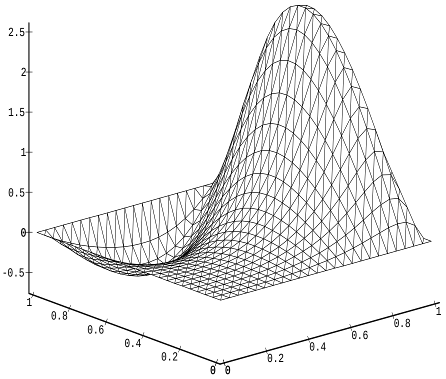


FIG. 3. *Plots of the exact solution $u(x)$*

Starting from the coarsest triangulation $\mathcal{T}_1$ (see Figure 4) the finer triangulations have been generated by dividing all triangles of the triangulation $\mathcal{T}_k$, $k = 1, 2, \ldots, l-1$, into four smaller congruent sub-triangles. In Table 2 we give the numbers of nodes and the numbers of triangles in each triangulation.

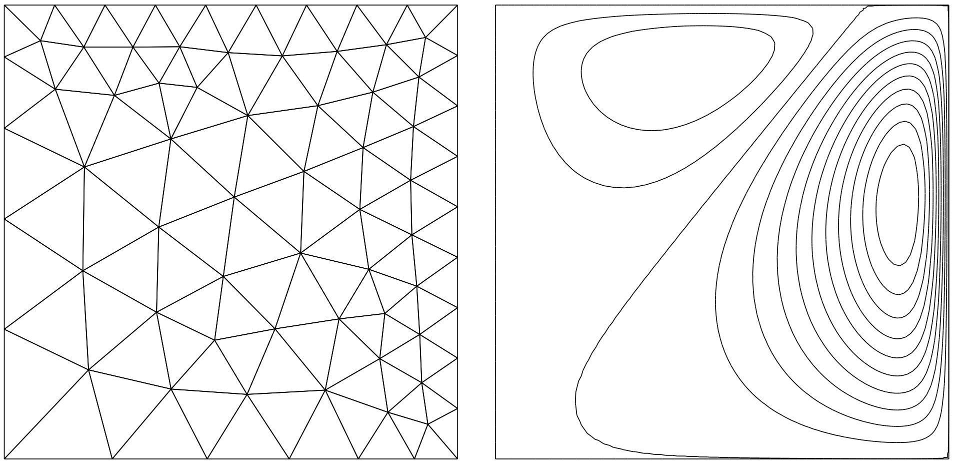


FIG. 4. *Mesh $\mathcal{T}_1$ and iso-lines of the solution u*

TABLE 2
Number of nodes and number of triangles in $\mathcal{T}_k$, $k = 1, 2, \ldots, 5$

triangulation	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$
number of nodes	75	268	1011	3925	15465
number of triangles	119	476	1904	7616	30464

For Algorithm MG–EX we used as pre-smoother two sweeps of the lexicographically forward Gauss-Seidel method for solving system (8), one iteration step of a $(l-1)$–grid algorithm for solving the coarse-grid system (11), and two sweeps of the lexicographically backward Gauss-Seidel method in the post-smoothing step. The initial guess was obtained by a full multigrid strategy. On the levels $k = 1, 2, \ldots, l-1$ a usual multigrid algorithm for solving the corresponding FE equations in the linear nodal basis was performed. Within this k–grid algorithm one V-cycle with two Gauss-Seidel sweeps lexicographically forward in the pre-smoothing step and two Gauss-Seidel sweeps lexicographically backward in the post-smoothing step were used. The convergence criterion for MG–EX was

$$\|\underline{f}_l^{L,ex} - K_l^{L,ex}\underline{u}_l^{(k+1,0)}\| \leq 10^{-4}\,\|\underline{f}_l^{L,ex} - K_l^{L,ex}\underline{u}_l^{(0,0)}\|\,, \tag{45}$$

where $\|\,.\,\|$ denotes the Euclidean norm in the space R^{N_l}, and $\underline{u}_l^{(0,0)}$ is the initial guess.

In Table 3 we present the number of iterations and the CPU-time needed by the application of the algorithm MG–EX. We obtain an improvement of the convergence behavior of our algorithm by introducing additional pre-smoothing and post-smoothing steps, i.e. before step 1 in the algorithm MG–EX we perform one iteration step of the Gauss-Seidel method lexicographically forward and after step 3 one iteration step of the Gauss-Seidel method lexicographically backward applied to the system of algebraic equations $K_l^{L,ex}\underline{u}_l^{L,ex} = \underline{f}_l^{L,ex}$. This is illustrated in column MG–EX(1) of Table 3.

TABLE 3
Comparison of the algorithm MG–EX and the algorithm MG–EX(1)

l	Algorithm MG–EX		Algorithm MG–EX(1)	
	number of iterations	CPU-time	number of iterations	CPU-time
3	13	2.42 sec	6	1.60 sec
4	13	11.31 sec	6	7.58 sec
5	14	52.46 sec	6	33.49 sec

Finally, we compare the discretization errors $\|u - u_l^{L,ex}\|$ and $\|u - u_l^Q\|$ in the H^1-norm and in the L_2-norm. Here $u_l^{L,ex}$ denotes the FE solution obtained by algorithm MG–EX (interpreted as a piecewise quadratic function) and u_l^Q the FE solution obtained directly by a discretization with piecewise p-hierarchical functions.

TABLE 4
Comparison of the discretization errors

Level l	$\|u - u_l^{L,ex}\|_{H^1}$	$\|u - u_l^{L,ex}\|_{L_2}$	$\|u - u_l^Q\|_{H^1}$	$\|u - u_l^Q\|_{L_2}$
3	0.23541-00	0.17906-02	0.23541-00	0.17906-02
4	0.59806-01	0.22421-03	0.59806-01	0.22421-03
5	0.15007-01	0.26197-04	0.15007-01	0.26198-04

Table 4 shows that the algorithm MG–EX yields discretization errors which are typical for discretizations with piecewise quadratic functions, i.e. we can observe an error of order $O(h_l^2)$ in the H^1-norm and $O(h_l^3)$ in the L_2-norm.

5. Conclusion. In this paper we have presented the analysis of an algorithm which can be understood algebraically as multigrid with τ-extrapolation. In practice, this algorithm is simple to implement, once a multigrid algorithm is available. However, we have shown that the algorithm converges to the same solution as a higher order finite element discretization. The algorithm can thus be used on unstructured meshes in an adaptive refinement setting. Furthermore, it is independent of global error expansions, and can thus be applied locally.

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